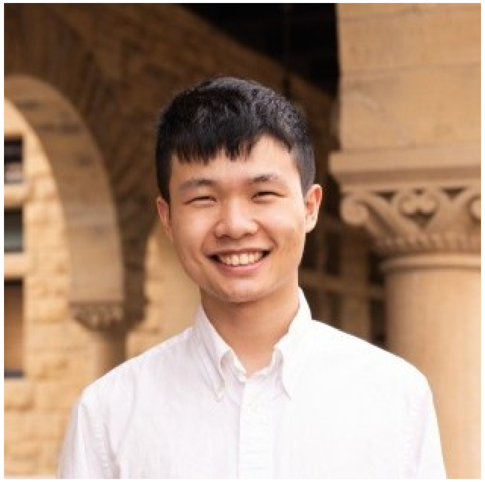
First-Order Logic

Part Two

The (Bonus) Teaching Team



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(Guest Instructor)

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Recap from Last Time

What is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - ***predicates*** that describe properties of objects,
  - ***functions*** that map objects to one another, and
  - ***quantifiers*** that allow us to reason about many objects at once.

Some spider is radioactive.

$$\exists s.\ (Spider(s) \land Radioactive(s))$$

∃ is the ***existential quantifier*** and says "for some choice of s, the following is true."

“Some *A* is a *B*”

translates as

$$\exists x.\ (A(x) \land B(x))$$

Useful Intuition:

Existentially-quantified statements are false unless there's a positive example.

$$\exists x.\ (A(x) \land B(x))$$

If x is an example, it must have property A on top of property B.

"For any natural number n,
n is even if and only if n^2 is even"

$$\forall n.\ (n \in \mathbb{N} \to (Even(n) \leftrightarrow Even(n^2)))$$

∀ is the ***universal quantifier*** and says "for any choice of n, the following is true."

“All *A*'s are *B*'s”

translates as

$$\forall x.\ (A(x) \to B(x))$$

Useful Intuition:

Universally-quantified statements are true unless there's a counterexample.

$$\forall x.\ (A(x) \to B(x))$$

If x is a counterexample, it *must* have property A but not have property B.

New Stuff!

The Aristotelian Forms

"All *A*s are *B*s"

$\forall x.\ (A(x) \to B(x))$

"Some *A*s are *B*s"

$\exists x.\ (A(x) \land B(x))$

"No *A*s are *B*s"

$\forall x.\ (A(x) \to \neg B(x))$

"Some *A*s aren't *B*s"

$\exists x.\ (A(x) \land \neg B(x))$

It is worth committing these patterns to memory. We'll be using them throughout the day and they form the backbone of many first-order logic translations.

The Art of Translation

Using the predicates

- *Person*(*p*), which states that *p* is a person, and
- *Loves*(*x*, *y*), which states that *x* loves *y*,

write a sentence in first-order logic that means “every person loves someone else.”

Question: How many of the following first-order logic statements are correct translations of "**everyone loves someone else**"?

Respond at pollev.com/cs103

$$\forall p.\ (Person(p) \to \exists q.\ (Person(q) \land Loves(p, q)))$$

$$\forall p.\ (Person(p) \land \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

$$\forall p.\ (Person(p) \to \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

$$\exists p.\ (Person(p) \land \forall q.\ (Person(q) \land p \neq q \to Loves(q, p)))$$

Every person loves someone else

Every person loves some other person

Every person p loves some other person

Every person p loves some other person

“All As are Bs”

$$\forall x.\ (A(x) \to B(x))$$

$\forall p. \ (Person(p) \to$

p loves some other person

$)$

"All As are Bs"

$\boldsymbol{\forall x. \ (A(x) \to B(x))}$

$\forall p. \ (Person(p) \rightarrow$

p loves some other person

$)$

$\forall p. \; (Person(p) \rightarrow$

there is some other person that p loves

$)$

$\forall p.\ (Person(p) \rightarrow$

there is a person other than p that p loves

$)$

$\forall p.\ (Person(p) \rightarrow$

there is a person q, other than p, where p loves q

$)$

$\forall p.\ (Person(p) \rightarrow$

there is a person q, other than p, where

p loves q

$)$

$\forall p.\ (Person(p) \to$

there is a person q, other than p, where
p loves q

$)$

> "Some *A*s are *B*s"
>
> $\boldsymbol{\exists x.\ (A(x) \land B(x))}$

$\forall p.\ (Person(p) \rightarrow$

$\exists q.\ (Person(q)\ \wedge$ *, other than p, where*

p loves q

$)$

$)$

"Some *A*s are *B*s"

$\boldsymbol{\exists x.\ (A(x)\ \wedge\ B(x))}$

$\forall p. \ (Person(p) \rightarrow$

$\exists q. \ (Person(q) \ \wedge$*, other than p, where*

p loves q

)

)

$\forall p.\ (Person(p) \rightarrow$

$\quad \exists q.\ (Person(q) \wedge p \neq q \wedge$

p loves q

$\quad)$

$)$

$$
\begin{aligned}
&\forall p.\ (Person(p) \to \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\quad\quad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

Using the predicates

- *Person*(p), which states that p is a person, and
- *Loves*(x, y), which states that x loves y,

write a sentence in first-order logic that means “there is a person that everyone else loves.”

There is a person that everyone else loves

There is a person p where everyone else loves p

There is a person p where everyone else loves p

“Some *A*s are *B*s”

$$\exists x.\ (A(x) \land B(x))$$

$\exists p.\ (Person(p)\ \land$

everyone else loves p

$)$

"Some *A*s are *B*s"

$\boldsymbol{\exists x.\ (A(x)\ \land\ B(x))}$

$\exists p.\ (Person(p)\ \wedge$

everyone else loves p

$)$

$\exists p. \; (Person(p) \; \wedge$

every other person q loves p

$)$

$\exists p.\ (Person(p)\ \wedge$

every person q, other than p, loves p

$)$

$\exists p.\ (Person(p) \land$

every person q, other than p, loves p

$)$

"All As are Bs"

$\forall \boldsymbol{x}.\ (\boldsymbol{A}(\boldsymbol{x}) \rightarrow \boldsymbol{B}(\boldsymbol{x}))$

$\exists p.\ (Person(p)\ \land$

$\quad \forall q.\ (Person(q)\ \land\ p \neq q \rightarrow$

q loves p

$\quad)$

$)$

"All *A*s are *B*s"

$\boldsymbol{\forall x.\ (A(x) \rightarrow B(x))}$

$\exists p.\ (Person(p)\ \land$

$\quad \forall q.\ (Person(q)\ \land\ p \neq q \rightarrow$

q loves p

$\quad)$

$)$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \forall q.\ (Person(q)\ \land\ p \neq q \to \\
&\qquad Loves(q, p) \\
&\quad) \\
&)
\end{aligned}
$$

$$\forall p.\ (Person(p) \rightarrow \exists q.\ (Person(q) \land Loves(p, q)))$$

$$\forall p.\ (Person(p) \land \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

$$\forall p.\ (Person(p) \rightarrow \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

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$$\exists p.\ (Person(p) \land \forall q.\ (Person(q) \land p \neq q \to Loves(q, p)))$$

Everyone loves someone else

$$\forall p.\ (Person(p) \rightarrow \exists q.\ (Person(q) \land Loves(p, q)))$$

$$\forall p.\ (Person(p) \land \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

$$\forall p.\ (Person(p) \rightarrow \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

Everyone loves someone else

$$\exists p.\ (Person(p) \land \forall q.\ (Person(q) \land p \neq q \rightarrow Loves(q, p)))$$

There is a person that everyone else loves

$$\forall p.\ (Person(p) \rightarrow \exists q.\ (Person(q) \land Loves(p, q)))$$

Everyone loves someone

$$\forall p.\ (Person(p) \land \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

$$\forall p.\ (Person(p) \rightarrow \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

Everyone loves someone else

$$\exists p.\ (Person(p) \land \forall q.\ (Person(q) \land p \neq q \rightarrow Loves(q, p)))$$

There is a person that everyone else loves

$$\forall p.\ (Person(p) \rightarrow \exists q.\ (Person(q) \land Loves(p, q)))$$

Everyone loves someone

$$\forall p.\ (Person(p) \land \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

Everything is a person and and loves someone else

$$\forall p.\ (Person(p) \rightarrow \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

Everyone loves someone else

$$\exists p.\ (Person(p) \land \forall q.\ (Person(q) \land p \neq q \rightarrow Loves(q, p)))$$

There is a person that everyone else loves

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: "Every person loves someone else"

For every person… $\forall p. \; (Person(p) \rightarrow$

… there is another person $\exists q. \; (Person(q) \land p \neq q \land$

… … they love $Loves(p, q)$

$)$

$)$

$$\forall p. \; (Person(p) \rightarrow \exists q. \; (Person(q) \land p \neq q \land Loves(p, q)))$$

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: "There is someone everyone else loves."

There is a person…	$\exists p.\ (Person(p)\ \land$
… that everyone else …	$\forall q.\ (Person(q)\ \land\ p \neq q \rightarrow$
… loves.	$Loves(q, p)$
	$)$
	$)$

For Comparison

For every person… $\forall p.\ (Person(p) \rightarrow$

… there is another person $\exists q.\ (Person(q) \land p \neq q \land$

… they love $Loves(p, q)$

$)$

$)$

There is a person… $\exists p.\ (Person(p) \land$

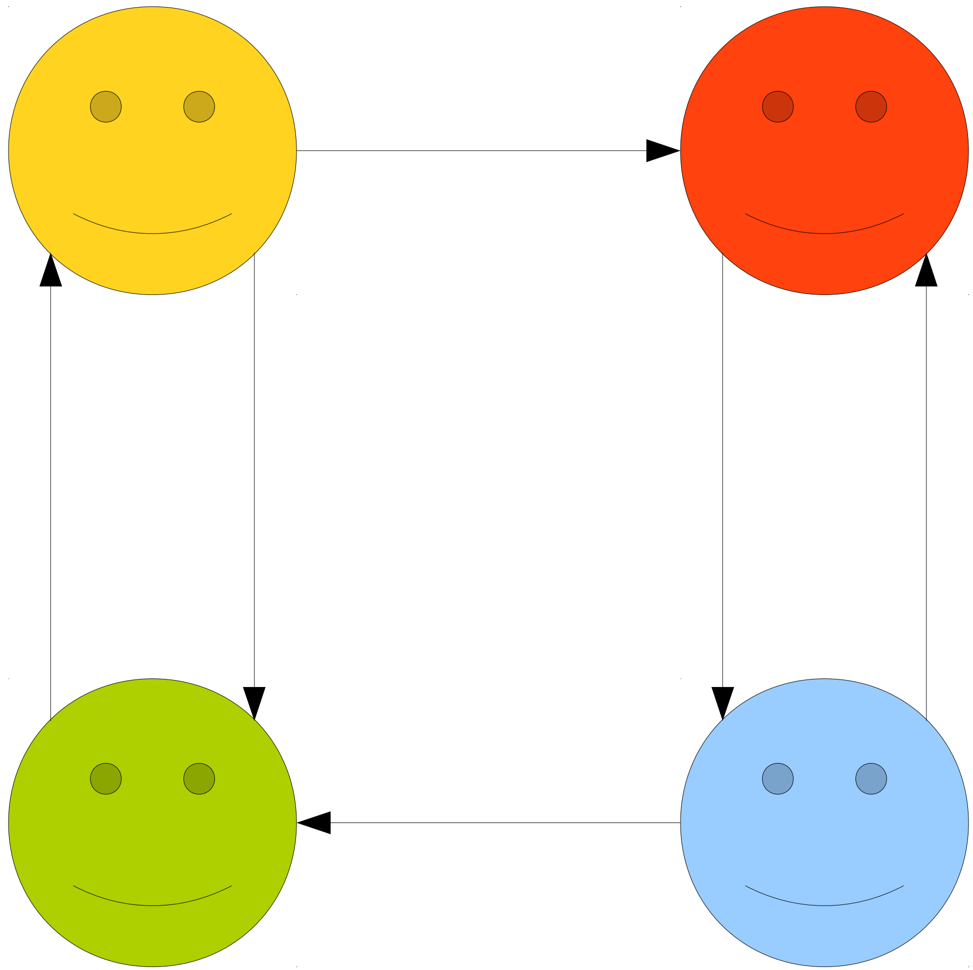
… that everyone else … $\forall q.\ (Person(q) \land p \neq q \rightarrow$

… loves. $Loves(q, p)$

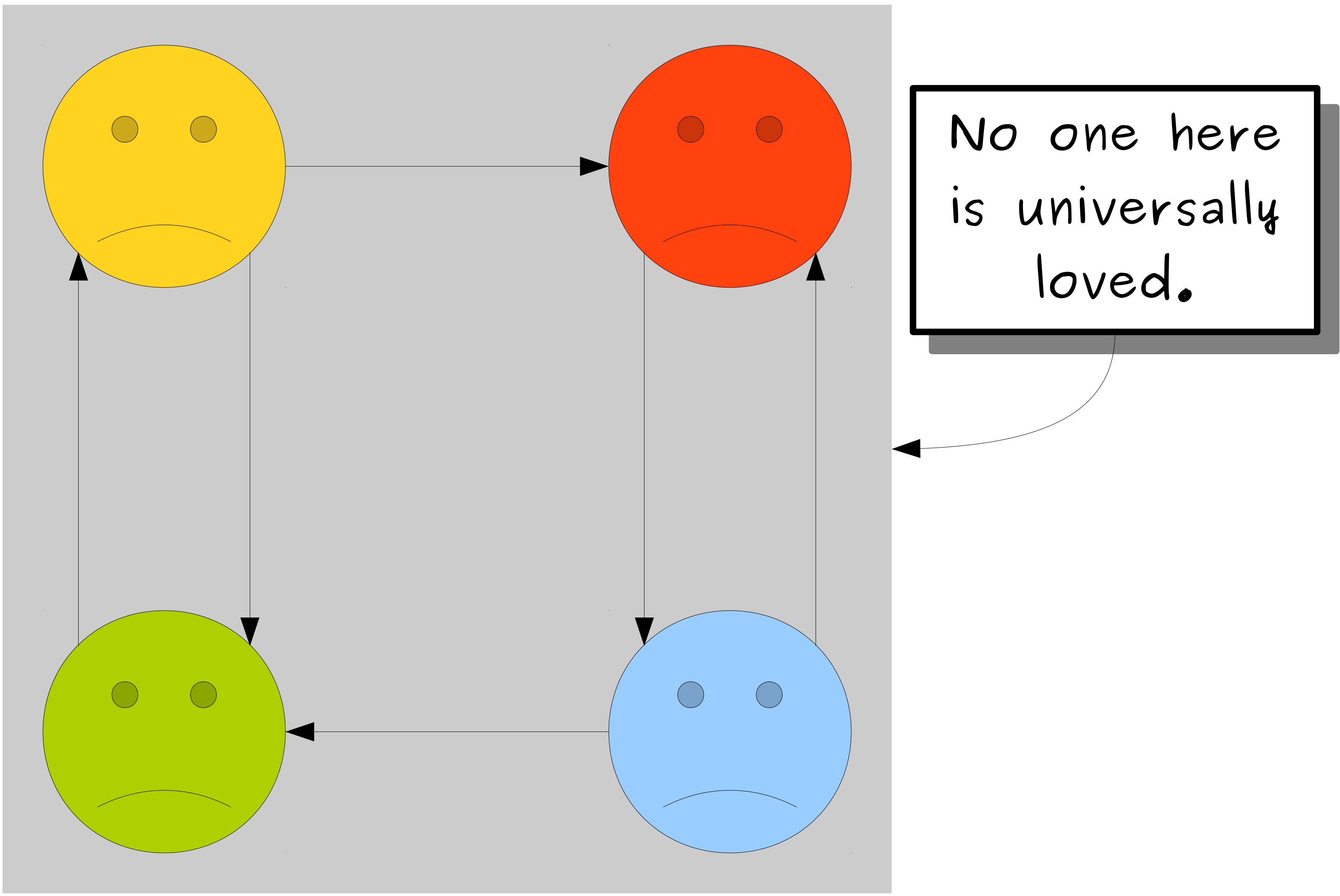
$)$

$)$

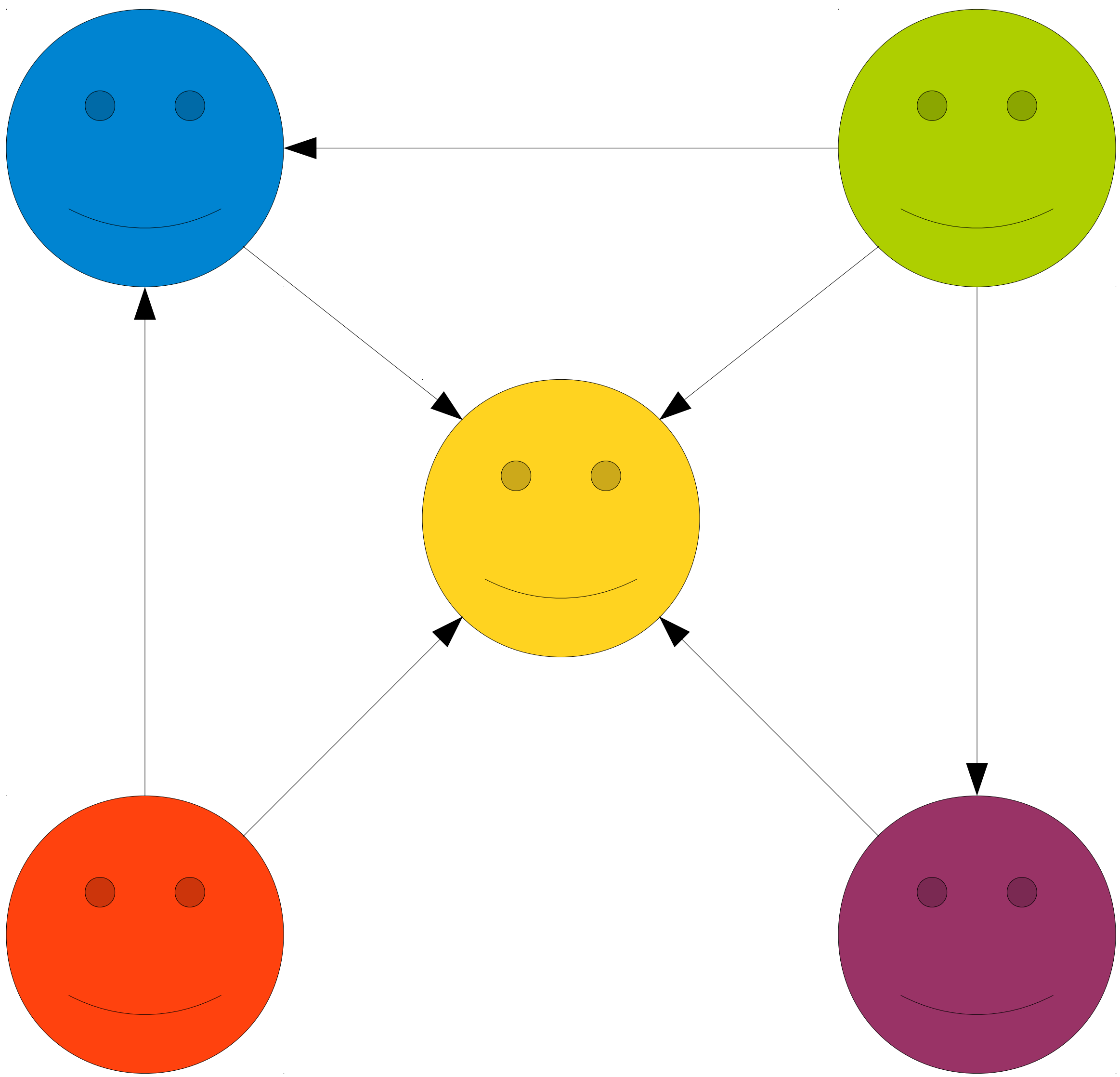
Every Person Loves Someone Else



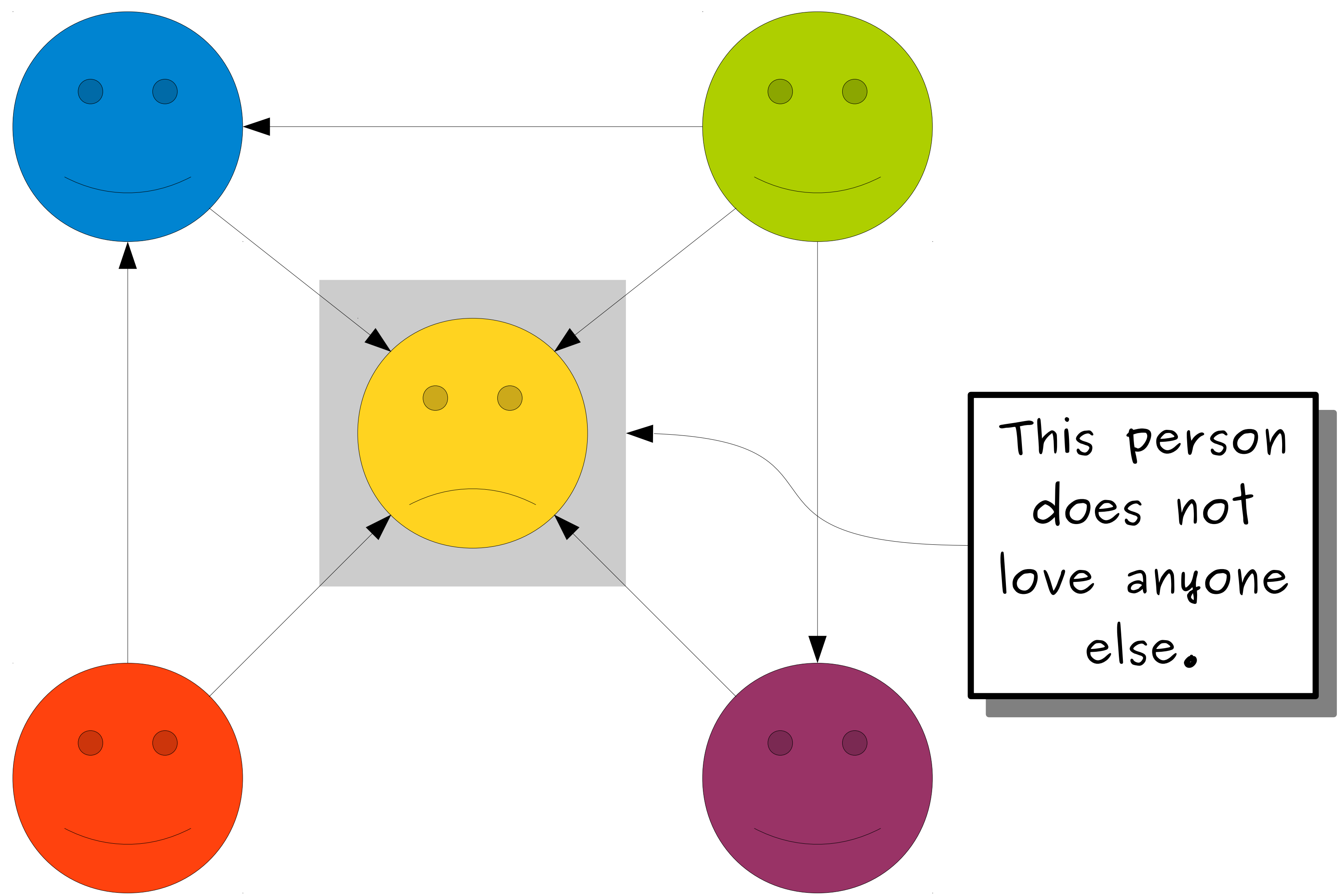
Every Person Loves Someone Else



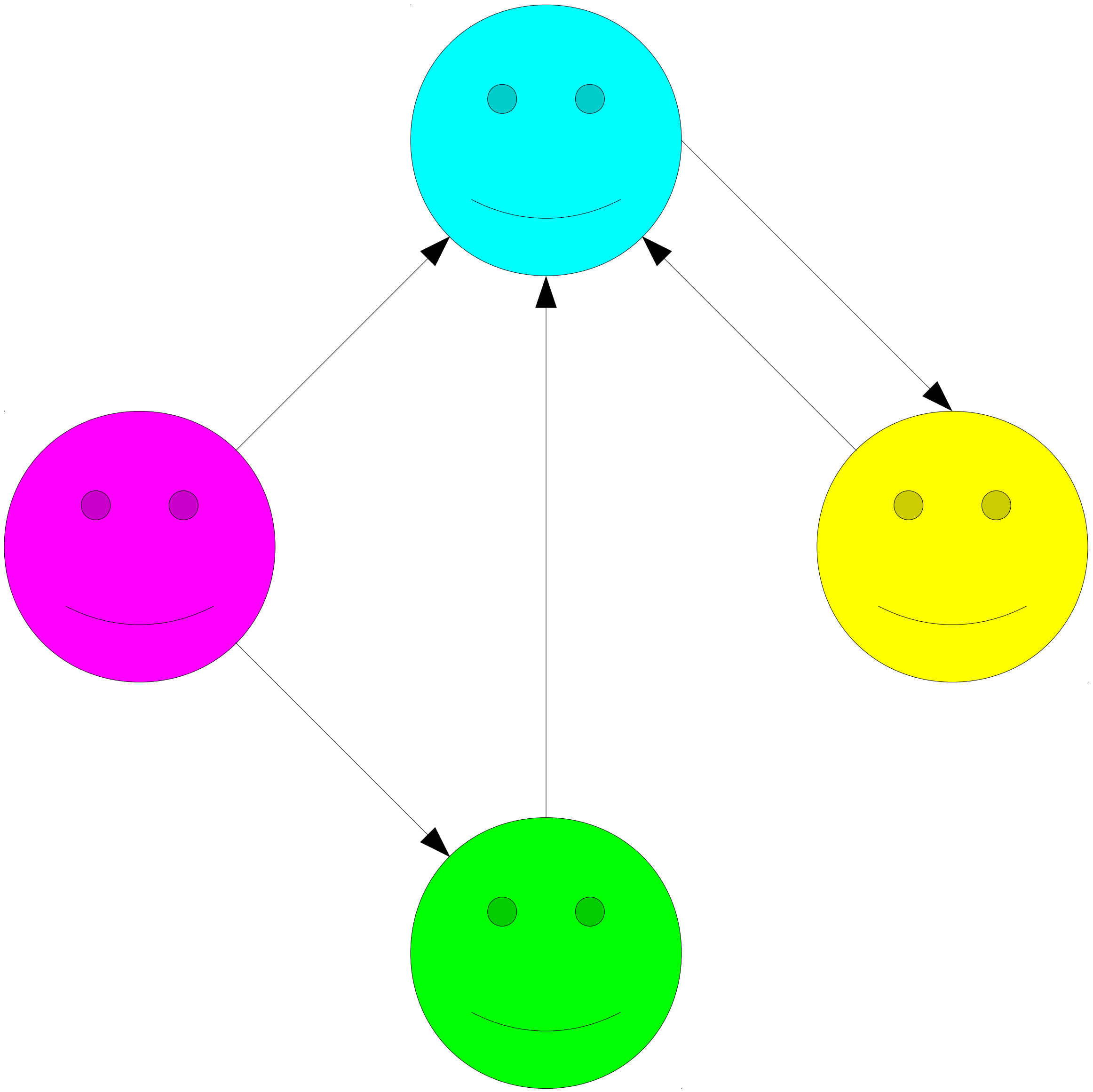
There is Someone Everyone Else Loves



There is Someone Everyone Else Loves



Every Person Loves Someone Else ***and*** There is Someone Everyone Else Loves



For every person… $\forall p. \, (Person(p) \rightarrow$

… there is another person $\exists q. \, (Person(q) \land p \neq q \land$

… … they love $Loves(p, q)$

$)$

$)$

and $\land$

There is a person… $\exists p. \, (Person(p) \land$

… that everyone else … $\forall q. \, (Person(q) \land p \neq q \rightarrow$

… loves. $Loves(q, p)$

$)$

$)$

Quantifier Ordering

- The statement

$$\forall x.\ \exists y.\ P(x, y)$$

means "for any choice of x, there's some choice of y where $P(x, y)$ is true."
- The choice of y can be different every time and can depend on x.

Quantifier Ordering

- The statement

  $$\exists x.\ \forall y.\ P(x, y)$$

  means "there is some *x* where for any choice of *y*, we get that *P*(*x*, *y*) is true."
- Since the inner part has to work for any choice of *y*, this places a lot of constraints on what *x* can be.

Order matters when mixing existential and universal quantifiers!

Time-Out for Announcements!

Problem Set One

- Problem Set One was due today at 4:00PM.
  - Didn't submit by then? Ping us ASAP.
- Reminder: partners should make a *single* Gradescope submission and make sure to add both partners to the submission group.
- Reminder: tag your pages on Gradescope!
  - (Can do either of these things after the deadline if you forgot.)

Problem Set Two

- Problem Set Two goes out today. It's due next Friday at 4:00PM.
  - Explore first-order logic, and expand your proofwriting repertoire.
- We have some online readings for this problem set.
  - Check out the ***Guide to Logic Translations*** for more on how to convert from English to FOL.
  - Check out the ***Guide to Negations*** for information about how to negate formulas.
  - Check out the ***First-Order Translation Checklist*** for details on how to check your work.

Back to CS103!

Set Translations

Using the predicates

- $Set(S)$, which states that S is a set, and
- $x \in y$, which states that x is an element of y,

write a sentence in first-order logic that means “the empty set exists.”

Using the predicates

- $Set(S)$, which states that S is a set, and
- $x \in y$, which states that x is an element of y,

write a sentence in first-order logic that means "the empty set exists."

First-order logic doesn't have set operators or symbols "built in." If we only have the predicates given above, how might we describe this?

·***Question:*** How many of the following first-order logic statements are correct translations of "the empty set exists"?

Respond at pollev.com/cs103

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \notin S)$$

$$\exists S.\ (Set(S) \land \neg \forall x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \neg \forall x.\ x \notin S)$$

The empty set exists.

There is some set S that is empty.

$\exists S. \ (Set(S) \ \wedge$

S is empty.

$)$

$\exists S.\ (Set(S)\ \wedge$

there are no elements in S

$)$

$\exists S. \ (Set(S) \ \land$

$\neg$ *there is an element in S*

$)$

$\exists S.\ (Set(S)\ \wedge$

$\neg$*there is an element x in S*

$)$

$$
\begin{aligned}
&\exists S.\ (Set(S)\ \land \\
&\quad \neg\exists x.\ x \in S \\
&)
\end{aligned}
$$

$$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$$

$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$

$\exists S.\ (Set(S) \land$

there are no elements in S

$)$

$\exists S.\ (Set(S) \wedge \neg \exists x.\ x \in S)$

$\exists S.\ (Set(S) \wedge$

every object does not belong to S

$)$

$\exists S.\ (Set(S) \wedge \neg \exists x.\ x \in S)$

$\exists S.\ (Set(S) \wedge$

every object x does not belong to S

$)$

$$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land$$
$$\quad \forall x.\ x \notin S$$
$$)$$

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

Both of these translations are correct. Just like in propositional logic, there are many different equivalent ways of expressing the same statement in first-order logic.

$$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

$$\exists S. (Set(S) \land \neg \exists x. x \in S)$$

$$\exists S. (Set(S) \land \forall x. x \notin S)$$

Why can we switch which quantifier we're using here?

Mechanics: Negating Statements

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	For some choice of x, $\neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	For all choices of x, $\neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For all choices of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	**For some choice of x, $\neg P(x)$**
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$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	**For some choice of x, $\neg P(x)$**	For all choices of x, $P(x)$

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$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	**For some choice of x, $\neg P(x)$**	For all choices of x, $P(x)$

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$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For all choices of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	**For all choices of x, $\neg P(x)$**
$\forall x.\ \neg P(x)$	**For all choices of x, $\neg P(x)$**	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For all choices of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	**$\exists x.\ \neg P(x)$**
$\exists x.\ P(x)$	For some choice of x, $P(x)$	**$\forall x.\ \neg P(x)$**
$\forall x.\ \neg P(x)$	**For all choices of x, $\neg P(x)$**	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For all choices of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For all choices of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	**For some choice of x, $P(x)$**	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	**For some choice of x, $P(x)$**
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For all choices of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	**For some choice of x, $P(x)$**	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For all choices of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For all choices of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	**For all choices of *x*, *P*(*x*)**	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of *x*, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of *x*, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of *x*, $\neg P(x)$	**For all choices of *x*, *P*(*x*)**

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	**For all choices of x, $P(x)$**	$\exists x.\ \neg P(x)$
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$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	$\forall x.\ P(x)$

An Extremely Important Table

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$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	$\forall x.\ P(x)$

Negating First-Order Statements

- Use the equivalences

  $$\neg\forall x.\ A \quad \textit{is equivalent to} \quad \exists x.\ \neg A$$

  $$\neg\exists x.\ A \quad \textit{is equivalent to} \quad \forall x.\ \neg A$$

  to negate quantifiers.
- Mechanically:
  - Push the negation across the quantifier.
  - Change the quantifier from ∀ to ∃ or vice-versa.
- Use techniques from propositional logic to negate connectives.

Taking a Negation

$$\forall x.\ \exists y.\ Loves(x, y)$$

("Everyone loves someone.")

$$\neg\forall x.\ \exists y.\ Loves(x, y)$$

$$\exists x.\ \neg\exists y.\ Loves(x, y)$$

$$\exists x.\ \forall y.\ \neg Loves(x, y)$$

("There's someone who doesn't love anyone.")

Two Useful Equivalences

- The following equivalences are useful when negating statements in first-order logic:

$$\neg(p \land q) \quad \textit{is equivalent to} \quad p \to \neg q$$

$$\neg(p \to q) \quad \textit{is equivalent to} \quad p \land \neg q$$

- These identities are useful when negating statements involving quantifiers.
  - $\land$ is used in existentially-quantified statements.
  - $\to$ is used in universally-quantified statements.
- When pushing negations across quantifiers, we *strongly recommend* using the above equivalences to keep $\to$ with $\forall$ and $\land$ with $\exists$.

Negating Quantifiers

- What is the negation of the following statement, which says "there is a cute puppy"?

$$\exists x.\ (Puppy(x) \land Cute(x))$$

- We can obtain it as follows:

$$\neg\exists x.\ (Puppy(x) \land Cute(x))$$

$$\forall x.\ \neg(Puppy(x) \land Cute(x))$$

$$\forall x.\ (Puppy(x) \to \neg Cute(x))$$

- This says "no puppy is cute."
- Do you see why this is the negation of the original statement from both an intuitive and formal perspective?

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

(*"There is a set with no elements."*)

$$\neg\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$
$$\forall S.\ \neg(Set(S) \land \forall x.\ \neg(x \in S))$$
$$\forall S.\ (Set(S) \to \neg\forall x.\ \neg(x \in S))$$
$$\forall S.\ (Set(S) \to \exists x.\ \neg\neg(x \in S))$$
$$\forall S.\ (Set(S) \to \exists x.\ x \in S)$$

(*"Every set contains at least one element."*)

Restricted Quantifiers

Quantifying Over Sets

- The notation

$$\forall x \in S.\ P(x)$$

  means "for any element *x* of set *S*, *P*(*x*) holds." (It's vacuously true if *S* is empty.)

- The notation

$$\exists x \in S.\ P(x)$$

  means "there is an element *x* of set *S* where *P*(*x*) holds." (It's false if *S* is empty.)

Quantifying Over Sets

- The syntax

$$\forall x \in S.\ P(x)$$

$$\exists x \in S.\ P(x)$$

  is allowed for quantifying over sets.
- In CS103, feel free to use these restricted quantifiers, but please do not use variants of this syntax.
- For example, don't do things like this:

⚠ $\forall x$ with $P(x).\ Q(x)$ ⚠

⚠ $\forall y$ such that $P(y) \land Q(y).\ R(y).$ ⚠

⚠ $\exists P(x).\ Q(x)$ ⚠

Expressing Uniqueness

Using the predicate

- *WayToFindOut*(w), which states that w is a way to find out,

write a sentence in first-order logic that means “there is only one way to find out.”

There is only one way to find out.

Something is a way to find out, and nothing else is.

Some thing w is a way to find out, and nothing else is.

Some thing w is a way to find out, and nothing besides w is a way to find out

$\exists w. \ (WayToFindOut(w) \ \wedge$

nothing besides w is way to find out

$)$

$\exists w.\ (\mathit{WayToFindOut}(w)\ \land$

anything that isn't w isn't a way to find out

$)$

$\exists w.\ (WayToFindOut(w)\ \wedge$

any thing x that isn't w isn't a way to find out

$)$

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w)\ \land \\
&\quad \forall x.\ (x \neq w \rightarrow \textit{x isn't a way to find out}) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w)\ \wedge \\
&\quad\ \forall x.\ (x \neq w \rightarrow \neg \mathit{WayToFindOut}(x)) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w)\ \land \\
&\quad \forall x.\ (x \neq w \rightarrow \neg \mathit{WayToFindOut}(x)) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w)\ \land \\
&\quad \forall x.\ (\mathit{WayToFindOut}(x) \rightarrow x = w) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w)\ \land \\
&\quad \forall x.\ (\mathit{WayToFindOut}(x) \rightarrow x = w) \\
&)
\end{aligned}
$$

Expressing Uniqueness

- To express the idea that there is exactly one object with some property, we write that
  - there exists at least one object with that property, and that
  - there are no other objects with that property.
- You sometimes see a special "uniqueness quantifier" used to express this:

$$\exists! x.\ P(x)$$

- For the purposes of CS103, please do not use this quantifier. We want to give you more practice using the regular ∀ and ∃ quantifiers.

Next Time

- ***Functions***
  - How do we model transformations and pairings?
- ***First-Order Definitions***
  - Where does first-order logic come into all of this?
- ***Proofs with Definitions***
  - How does first-order logic interact with proofs?